

# Higgs potential and confinement in Yang-Mills theory on exotic $\mathbb{R}^4$

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We show that pure  $SU(2)$  Yang-Mills theory formulated on certain exotic  $\mathbb{R}^4$  from the radial family shows confinement. The condensation of magnetic monopoles and the qualitative form of the Higgs potential are derived from the exotic  $\mathbb{R}^4$ ,  $e$ . A relation between the Higgs potential and inflation is discussed. Then we obtain a formula for the Higgs mass and discuss a particular smoothness structure so that the Higgs mass agrees with the experimental value. The singularity in the effective dual  $U(1)$  potential has its cause by the exotic 4-geometry and agrees with the singularity in the maximal abelian gauge scenario. We will describe the Yang-Mills theory on  $e$  in some limit as the abelian-projected effective gauge theory on the standard  $\mathbb{R}^4$ . Similar results can be derived for  $SU(3)$  Yang-Mills theory on an exotic  $\mathbb{R}^4$  provided dual diagonal effective gauge bosons propagate in the exotic 4-geometry.

## I. INTRODUCTION

The first indication that Dirac magnetic monopoles might be related with certain small exotic  $\mathbb{R}^4$ s from the fixed radial family, appeared in [1, 2]. The derivation was based on the algebraic agreement between the magnetic charges of Dirac monopoles and the Godbillon-Vey (GV) classes of certain foliations generating small smooth exotic  $\mathbb{R}^4$ s grouped in the radial family. In our recent work [3] we pushed forward the approach and showed that in the radial family there exists (at least one) exotic  $e$  representing the geometry of the BPS monopoles moduli space,  $\mathcal{M}_2^0$ , where the Higgs potential vanishes. When Yang-Mills (YM) theory is formulated on  $e$  and when the smooth structure on  $e$  is reduced to the standard one, then Polyakov-'t Hooft BPS monopoles of the 3-dimensional Yang-Mills-Higgs theory appear as an assignment of the exotic  $\mathbb{R}^4$ ,  $e$ . This assignment was possible by considering the special foliated topological limit (FTL) of general relativity (GR) on  $e$ , so that quasi-modularity of the metrics became important. However, the geometry of  $\mathcal{M}_2^0$  is not the exotic geometry of the end of  $e$ . It is just the 'semi-classical' approximation where the gravitational Euclidean path integral on  $e$  is considered and where  $\mathcal{M}_2^0$  contributes as gravitational instanton [3].

In this paper we extend this approach to the general non-zero Higgs potential emerging from a full exotic 4-geometry on  $\mathbb{R}^4$ . The precise shape of the Higgs potential is obtained as the Morse function describing the exotic handle-body structure. In contrast to our previous work, we will describe the end of the small exotic  $\mathbb{R}^4$ ,  $e$ , which is an exotic  $S^3 \times_\theta \mathbb{R}$ . In this way the BPS condition, hence zero Higgs potential, corresponds merely to the 4-geometry, i.e. Atiyah-Hitchin gravitational instanton, whereas the general non-zero Higgs potential corresponds to the true exotic handle-bodies with non-canceling (smoothly) pairs of handles. Furthermore, by assuming a connection to inflation and choosing a special exotic  $S^3 \times_\theta \mathbb{R}$ , we are able to calculate the Higgs mass which is in a good agreement with the experimental value.

Thus, given the YM theory on exotic  $e$  the Higgs field can be introduced from this exotic 4-geometry, and the Yang-Mills-Higgs (YMH) theory with the general Higgs potential can be correctly described. In this way we would have a kind of mechanism for geometric confinement in the  $SU(2)$  YM theory on an exotic  $\mathbb{R}^4$ . Magnetic monopoles and the Higgs field are now rather ingredients of the exotic 4-geometry than in pure YM on the standard  $\mathbb{R}^4$ . That is why a more direct connection of (twisted) YM  $SU(2)$  theory without Higgs field and magnetic monopoles, with exotic 4-geometry should exist. In this paper we describe this connection in two steps, realizing a geometric confinement in the YM theory on an exotic  $\mathbb{R}^4$ .

1) YM theory is twisted so that it fits with the current and magnetic monopoles as well Higgs potential from exotic  $\mathbb{R}^4$ . The twisting is just the abelian (maximal) gauge and is the result of the asymmetric propagation of gauge fields (gluons). This is the place where exotic 4-geometry intervenes directly. The propagation relies on the choice that diagonal dual  $U(1)$  fields live on exotic  $\mathbb{R}^4$  while the electric field is not sensitive to exotic 4-geometry of the background and propagates in the standard smooth structure of  $\mathbb{R}^4$ . This asymmetry is the geometric reason for the abelian projection in YM theory. There exist various twisting versions of YM theory and we discuss here two of them:

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The maximal abelian gauge and abelian projected effective gauge theory (APEGT) [4, 5] theories. Both theories show confinement.

2) However, the exotic 4-geometry of  $e$  carries also the mechanism for breaking the dual magnetic  $U(1)$  symmetry (and the Bianchi identity). This is the singularity in the magnetic  $U(1)$  field due to the change of the smoothness from exotic to the standard one. The fixed radial family introduces the magnetic current into the YM theory which is nothing but the nontrivial Godbillon-Vey class of the associated codimension-1 foliation. The non-zero magnetic current, in turn, via the condensation of magnetic monopoles, generates the confinement in the YM theory.

This 2-step process indicates the very fundamental role played by exotic  $\mathbb{R}^4$ s, in particular the non-trivial 4-geometry which becomes an important ingredient to explain the confining/deconfining change of phases in YM theory.

We close the paper with the short discussion of the proposed geometric scenario for the confinement in  $SU(2)$  theory on  $e$ . Still, a more thorough understanding of the physical meaning of the geometric asymmetry in the gluon propagation is needed. This problem, along with the analysis of the physical case of  $SU(3)$  QCD on exotic  $\mathbb{R}^4$ , will be presented separately.

## II. 3-D BPS MAGNETIC MONOPOLES AND HIGGS FIELD IN YMH THEORY FROM EXOTIC $\mathbb{R}^4$

In this section we recapitulate some of the results obtained in [3]. The 4-d  $SU(2)$  YMH theory on  $\mathbb{R}^4$  with the Minkowski metric, is given by the density of the Lagrangian [6]:

$$\mathcal{L} = -\frac{1}{2}\text{Tr}(F_{\mu\nu}F^{\mu\nu}) + \text{Tr}(D^\mu\Phi D_\mu\Phi) - U(\Phi) = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - U(\Phi^a) \quad (1)$$

where  $U(\Phi) = -\frac{\lambda}{2}(\text{Tr}(\Phi^2) - v^2)^2 = \frac{\lambda}{4}(\Phi^a\Phi^a - v^2)^2$  is the gauge-invariant potential and  $a$  is the internal index in the  $su(2)$  algebra. One sees that the spectrum of this theory contains three massless vector bosons and a massive scalar Higgs field. Defining the fields  $E^{ai} = F^{ai0}$  and  $B^{ai} = -\frac{1}{2}\epsilon^{ijk}F_{ajk}$  in the usual way, the energy of the theory reads:

$$\mathcal{E} = \frac{1}{2}\{E_i^a E_i^a + B_i^a B_i^a + D_0\Phi^a D_0\Phi^a + D_i\Phi^a D_i\Phi^a\} + U(\Phi), \quad (2)$$

so that the vacuum configuration is given by  $A_\mu = 0$  and  $|\Phi| = v = \text{const}$  along the gauge transformations of this theory. Given the vacuum as  $A_\mu^v = 0$ ,  $\Phi^v = (0, 0, v)$  and choosing the gauge so that  $\phi = (0, 0, \phi_3)$  and  $\Phi = \Phi^v + \phi$ , the Lagrangian of the YMH theory in these new fields, reads:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \frac{(ev)^2}{2}A^{1\mu}A_\mu^1 + \frac{(ev)^2}{2}A^{2\mu}A_\mu^2 + \frac{1}{2}\partial^\mu\phi_3\partial_\mu\phi_3 - \frac{1}{2}\lambda v^2(\phi_3)^2. \quad (3)$$

The spectrum now contains two massive  $A_\mu^1, A_\mu^2$  and one massless  $A_\mu^3$  vector bosons and the massive  $\phi_3$  Higgs field with the classical mass  $v\sqrt{\lambda}$ . In this reformulated YMH theory one embeds the electromagnetism such that magnetic monopoles appear. This is performed via the gauge-invariant electromagnetic tensor found by 't Hooft:

$$f_{\mu\nu} = \text{Tr}(\hat{\Phi}F_{\mu\nu}) - \frac{1}{e}\text{Tr}(\hat{\Phi}[D_\mu\hat{\Phi}, D_\nu\hat{\Phi}]) = \partial_\mu(A_\nu^a\hat{\Phi}^a) - \partial_\nu(A_\mu^a\hat{\Phi}^a) - \frac{1}{e}\epsilon^{abc}\hat{\Phi}^a\partial_\mu\hat{\Phi}^b\partial_\nu\hat{\Phi}^c$$

where  $\hat{\Phi} = \frac{\Phi}{\text{Tr}(\Phi^2)}$ . New electric and magnetic fields as well as electric,  $j_\mu^e$ , and magnetic,  $j_\mu$ , currents, thus read:

$$\begin{aligned} E^i &= f^{i0}, B^i = -\frac{1}{2}\epsilon^{ijk}f_{jk} \\ j_\mu^e &= -\partial_\nu f_{\nu\mu}, j_\mu = \partial_\nu^* f_{\nu\mu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu f^{\rho\sigma} = -\frac{1}{2e}\epsilon_{\mu\nu\rho\sigma}\epsilon^{abc}\partial_\nu\hat{\Phi}^a\partial_\rho\hat{\Phi}^b\partial_\sigma\hat{\Phi}^c \end{aligned} \quad (4)$$

where it is seen that  $\partial^\mu j_\mu = 0$ . This electromagnetic theory exhibits electric-magnetic duality:

$$\begin{aligned} j_0 &= \nabla \cdot B, \quad j_i = -(\nabla \times E)_i - \partial_t B_i \\ j_0^e &= -\nabla \cdot E, \quad j_i^e = (\nabla \times B)_i - \partial_t E_i. \end{aligned}$$

The equations of motion derived from the Lagrangian of the YMH theory (YMH eqs.), read:

$$D_\mu F^{\mu\nu} = e[\Phi, D^\nu\Phi], \quad D^\mu D_\mu\Phi = -\lambda\Phi\text{Tr}(\Phi^2 - v^2).$$

In order to have finite energy configurations one has to have  $U(\Phi) \xrightarrow{x \rightarrow \infty} 0$  so that  $\Phi^a\Phi^a \xrightarrow{x \rightarrow \infty} v^2$ . Thus, the topological charge, the winding number  $N$  of the 2-sphere by the 2-sphere as element of the homotopy group  $\pi_2(S^2) = \mathbb{Z}$ , is related to the total magnetic charge  $g$  of the field configuration:

$$g = \frac{4\pi N}{e}.$$

However, to solve the YMH equations in full generality is the formidable task, and because of that one considers the special limit, namely the BPS limit, where  $U(\Phi)$  is zero in the YMH Lagrangian. Moreover, the BPS and static configurations are considered. In this way, the YMH theory reduces to 3-dimensions and the space of solutions is tractable and can be explicitly presented. This exactly solvable, though reduced to 3-d, case allows one to relate the geometry of the moduli space  $\mathcal{M}_2^0$  of the  $k = 2$  magnetic monopoles, with some exotic 4-geometry on  $\mathbb{R}^4$ . All small exotic  $\mathbb{R}^4$ s are considered as being grouped in the fixed radial family of such structures which happens to be crucial in derivation of the relation [3].

The above mentioned 3-d reduction of the 4-d YMH theory on Minkowski 4-space, can be also seen as the reduction of the 4-d pure YM theory on the Euclidean  $\mathbb{R}^4$ . The Higgs field appears in 3-d YMH theory from this 4-d YM theory though with the vanishing Higgs potential. Moreover, the self-duality of 4-d Euclidean pure YM theory enforces the theory to fulfill the BPS condition after reduction to 3-d. We would like to find a similar appearance of the Higgs field and magnetic monopoles also for a general Higgs potential. This potential is attempted to be obtained from an exotic  $\mathbb{R}^4$ , which the YM theory is formulated on.

First, let us recall how the 4-d to 3-d reduction looks like for the YM theory on Euclidean  $\mathbb{R}^4$ . The pure Euclidean  $SU(2)$  YM theory on  $\mathbb{R}^4$  is given by the action

$$S = -\frac{1}{2} \int d^4x \text{Tr}(F_{\mu\nu}^a F^{a\mu\nu}) \quad (5)$$

where as usual  $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{\beta\gamma}^a A_\mu^\beta A_\nu^\gamma$  with structure constants  $f_{\beta\gamma}^a$  of  $SU(2)$  and the vector potential  $A_\mu^a$ ,  $\mu = 1, 2, 3, 4$  takes values in the  $su(2)$  algebra. The dual tensor  ${}^*F$  to  $F$  is defined by  ${}^*F_{\mu\nu}^a = \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^a$ . The equations of motion (EOM) derived from (5) in the component-free notation, read:

$$D_A F = 0 = D_A {}^*F$$

where  $D_A = d + A$  is the covariant exterior derivative (depending on the connection  $A$ ). The (anti-)self-duality condition now reads  $F_{\mu\nu}^a = \mp {}^*F_{\mu\nu}^a$  and the connection  $A$  realizing this equation, solves automatically the above YM EOM since they reduce to the Bianchi equations.

Suppose now that  $A_4 = \phi$  such that  $A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3 + \phi dx_4$  where  $A_i, i = 1, 2, 3$  and  $\phi$  are  $su(2)$ -algebra valued functions on  $\mathbb{R}^3$ . Also, the YM EOM (5) are invariant with respect to the translations in the  $x_4$ -directions, meaning that the transformed and the original configurations are equivalent up to a gauge. The Euclidean YM Lagrangian for this reduced YM theory reads:

$$\mathcal{L} = -\frac{1}{2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) + \text{Tr}(D^\mu \phi D_\mu \phi) = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} D^\mu \phi^a D_\mu \phi^a. \quad (6)$$

where  $F$  and  $D_A$  are now defined with respect to the connection  $A_1 dx_1 + A_2 dx_2 + A_3 dx_3$  with the translational invariance in  $x_4$  of the expressions. In this way we have a 3-d YMH theory with  $\lambda = 0$ , so that the Higgs potential vanishes. The Bogomolny equations in this setting give:

$$B_i = D_i \phi = D_i A_4 = E_i$$

which means that the connection is self-dual. Thus, finally we have a 3-d YMH theory with vanishing Higgs potential fulfilling the Bogomolny equations, i.e. the static BPS monopoles configurations are described here, and this theory is the dimensional reduction of the pure self-dual Euclidean 4-d YM theory (without any Higgs field) [6, 7]. The quantum dynamics of these BPS monopoles, hence 4-d case, is partially restored (in the low energy limit and for slow motions) by considering the geodesic approximation on the space of moduli of the static BPS monopoles [8]. On the other hand, the geometry of the space of the moduli  $\mathcal{M}_2^0$  of the  $k = 2$  magnetic monopoles is related with the exotic 4-geometry on some fake 4-space from the radial family [3]. It was performed by considering the special *foliated topological limit* (FTL) of general relativity on exotic  $\mathbb{R}^4$ . In this limit quasi-modular, i.e. depending on the Eisenstein 2-nd series  $E_2$ , expressions become dominant and so, the geometry of  $\mathcal{M}_2^0$  does. However, it is still an approximation to the true exotic geometry of the end of exotic  $\mathbb{R}^4$ ,  $e$ . It is conjectured that precisely  $\mathcal{M}_2^0$ , being the gravitational instanton, gives the dominant contribution to the Euclidean path integral on the exotic end of  $e$  [3].

Thus the assignment of  $k = 2$  magnetic monopoles to exotic  $\mathbb{R}^4$  is rather effective and approximative. On the one hand this is due to the absence of Higgs potential in the BPS limit. On the other hand the reason is the argumentation based on semi-classical path integral. To approach confinement in such setting we would like to have rather the general Higgs potential and the reduction to the dual abelian Higgs theory. In the next section we are going to show that the shape of general Higgs potential follows from the exotic handle-body topology of  $e$ .

### III. GENERAL HIGGS POTENTIAL FROM OPEN EXOTIC 4-GEOMETRY

Let us consider the end of a small exotic  $\mathbb{R}^4$ , i.e. a sequence of compact (codim-0) subspaces  $K_1 \subset K_2 \subset K_3 \subset \dots$  leading to a sequence  $U_1 \subset U_2 \subset U_3 \subset \dots$  of complements  $U_i = \mathbb{R}^4 \setminus K_i$ . The end of every topological  $\mathbb{R}^4$  is

topologically  $S^3 \times \mathbb{R}$ . Furthermore, the end of an exotic [24]  $\mathbb{R}^4$  is an exotic  $S^3 \times \mathbb{R}$ . In the following we will discuss the appearance of the Higgs potential from the topology of the exotic  $S^3 \times_\theta \mathbb{R}$  as end of a small exotic  $\mathbb{R}^4$ . In contrast to our previous work where the exotic  $S^3 \times_\theta \mathbb{R}$  was simply constructed from the homology 3-sphere  $\Sigma$  appearing as the smooth cross-section [9, 10] we need now the full power of Casson handles. But surprisingly the result is very similar. At first we state a known result [11, 12] that  $n$ th untwisted Whitehead double of the pretzel knot (3,3,-3) (or the knot  $9_{46}$  in Rolfsen notation) separates the end ( $S^3 \times_\theta \mathbb{R}$ ) from the small exotic  $\mathbb{R}^4$ . But then we have to consider only the Casson handle  $CH$  in  $S^3 \times_\theta \mathbb{R}$  coming from a non-canceling pair of 1-/2-handles (see below for an explanation). By definition, this  $CH$  is exotic, i.e. the attaching circle is not smoothly slice. Bizaca (see for instance Corollary 0.3 in [13]) constructed a family of Casson handles fulfilling this property. The attaching circle is bounded by a knot. The idea of the proof is the construction of a knot so that any (finite)  $n$ -fold untwisted Whitehead double of this knot does not bound a smooth disk. The result is the right-handed (or positive) trefoil knot. Following [12], we constructed a smooth section in  $S^3 \times_\theta \mathbb{R}$  by Dehn surgery in dependence on the framing of the attaching circle. For the framing +1 we obtain the Poincare homology 3-sphere and a sequence of homology 3-spheres as Whitehead doubles of the trefoil knot. In particular, every other knot smoothly concordant to the trefoil knot (like the knot  $8_{10}$  see below) can be also used. But one thing remains: we obtain a (homology) cobordism (as a 4-manifold) in the exotic  $S^3 \times_\theta \mathbb{R}$  between  $S^3$  and a homology 3-sphere  $\Sigma$  (not bounding a contractable, smooth 4-manifold).

The Morse theory associated with the manifold  $\Sigma$  is determined by considering a scalar function  $f : \Sigma \rightarrow [0, 1]$  over  $\Sigma$ . The number of critical points of the Morse-function  $f$  are related to the dimension of the homology-groups of  $\Sigma$ . Let us consider two Morse-functions  $\psi : S^3 \rightarrow \mathbb{R}$  and  $\Psi : \Sigma \rightarrow \mathbb{R}$ . In the context of cosmological models one can consider them as corresponding to two cosmic states  $S^3$  and  $\Sigma$  in the exotic  $S^3 \times_\theta \mathbb{R}$  model where  $\Sigma$  is the homology 3-sphere constructed from the knot  $8_{10}$  (see [14] for an explanation of this choice). The spacetime and in particular its smoothness structure is represented by the homology cobordism  $\phi : \Sigma \rightarrow S^3$  (a cobordism preserving the homology groups) between these states. This mapping  $\phi$  factorizes the Morse-function  $\Psi$  on  $\Sigma$  in two maps  $\Psi = \psi \circ \phi$ , i.e. the diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{\Psi} & \mathbb{R} \\ \phi \downarrow & \circlearrowleft & \uparrow id \\ S^3 & \xrightarrow{\psi} & \mathbb{R} \end{array} \quad (7)$$

commutes. The new map  $\psi$  is a representation of  $\Psi$  on  $S^3$ . On the 3-sphere there exist two critical points: one maximum and one minimum. That means that the cobordism  $M$  possesses the homology groups  $H_0(M) = H_3(M) = \mathbb{Z}$  of a homology 3-sphere. The evaluation of the exact sequence of the pair  $(M, \partial M)$  shows that  $H_0(M)$  is generated by one of the boundary components, e.g. by  $H_0(S^3)$  and  $H_3(M)$  by the other one, i.e.  $H_3(\Sigma)$ . The transition  $y = \phi(x)$  represented by  $M$  maps the Morse function  $\psi(y) = ||y||^2$  on  $S^3$  to the Morse function  $\Psi(x) = ||\phi(x)||^2$ .

A field-theoretic discussion of the Morse-theoretic result starts with the following simple idea: The 3-sphere is isomorphic to  $SU(2)$  and so one can define a formal group-operation on  $S^3$ . This makes the map  $\phi$  to a  $SU(2)$ -valued scalar-field over  $\Sigma$ . Witten has derived the Morse-theory from the field-theoretic properties [23]. His ansatz gives a field-theoretic construction of the dynamics which gives a “tunneling path” between two critical points. In our case, the field of the Witten-construction is the  $SU(2)$ -scalar  $\phi$  over the homology 3-sphere  $\Sigma$ . Using his  $\sigma$ -model we get a field theory with the Lagrange density

$$\mathcal{L}_\xi = D_\mu \phi \cdot D^\mu \phi + \phi \cdot \phi,$$

with the covariant derivation  $D_\mu$  and a representation dependent product  $\cdot$  of the group  $SU(2)$ . Adding the Einstein-Hilbert-action we obtain the well known model of chaotic inflation

$$\mathcal{L} = \frac{1}{\kappa} R + D_\mu \phi \cdot D^\mu \phi + \frac{\rho_G}{2} \phi \cdot \phi, \quad (8)$$

so that the topological transition  $S^3 \rightarrow \Sigma$  of the cosmological model can be interpreted as the inflation of the cosmos [10]. The factor  $\rho_G$  can be interpreted as the curvature of the homology 3-sphere  $\Sigma$  represented as energy density. The Lagrangian (8) is only valid, if the cobordism is a smooth 4-manifold leading to a smooth transition  $S^3 \rightarrow \Sigma$ . But in the construction of the exotic  $S^3 \times_\theta \mathbb{R}$ , the end of an exotic  $\mathbb{R}^4$ , one needs a non-smooth cobordism together with a Casson handle reflecting the exoticness of  $S^3 \times_\theta \mathbb{R}$ . Casson handles are also responsible for the exotic smoothness on Euclidean  $\mathbb{R}^4$  which is our main concern in this paper. Namely, consider the construction of a homology cobordism, like  $\phi : \Sigma \rightarrow S^3$ , where Casson handles have to appear. The problem is the introduction of the 2-/3-handle pair. The existence of the exotic smoothness structure forbids the smooth cancellation of the handle pairs so that the infinite layer structure of the Casson handle appears. The generic case is given by the following model. Glue a 1-/2-handle pair (dual to a 2-/3-handle pair) to a 0-handle. The 0-handle is modeled by a minimum in Morse theory. But the 1-/2-handle pair generates a maximum and a minimum which can be canceled.

Exotic smoothness forbids this cancellation and we obtain an extra pair of maximum/minimum. In the notation above by using the commutative diagram (7), we can model a Morse function with two minima and one maxima by the Morse function

$$\psi(y) = ||y||^4 - ||y||^2 \quad (9)$$

on  $\Sigma$ . We describe the transition by a  $SU(2)$ -valued scalar field  $\phi : \Sigma \rightarrow SU(2)$  and obtain

$$\mathcal{L} = \frac{1}{\kappa} R + D_\mu \phi \cdot D^\mu \phi + \frac{\rho_G}{2} (\phi^4 - \phi^2), \quad (10)$$

as Lagrangian. By using Cerf theory [15], the choice of the function (9) is generic which we will explain now. The Morse function of the cobordism can be interpreted as a one-parameter family of Morse functions. Then, there three types of functions: the usual Morse function (as sum/difference of quadratic terms), the birth-death point as function  $x^3$  plus a Morse function and the dovetail function  $x^4$  plus a Morse function. Because of the 2/3- handle pair (codimension-2 case), we have to choose the dovetail (having a codimension-2 critical point). The unfolding (the one-parameter family) is  $x^4 \pm tx^2$  (up to a linear term and a Morse function). Therefore the function (9) is generic, i.e. every smooth deformation of the spaces in the diagram (7) do not change the form of this function. This unfolding of the function (9) has also a direct physical background: the deformation parameter can be interpreted as the mass of the Higgs field

$$\mathcal{L} = \frac{1}{\kappa} R + D_\mu \phi \cdot D^\mu \phi + \frac{\rho_G}{2} (\phi^4 - \phi^2) + \frac{M_H}{2} \phi^2.$$

But then by a careful study of the transition  $S^3 \rightarrow \Sigma$ , we should be able to predict the mass of the Higgs field. In [10], we studied this process and discussed its relation to inflation. In particular we obtained the scaling (or expansion) induced by this process to be

$$a = a_0 \cdot \exp\left(\frac{3 \cdot \text{vol}(\Sigma)}{2 \cdot CS(\Sigma)}\right) = L_P \cdot \exp\left(\frac{3 \cdot \text{vol}(\Sigma)}{2 \cdot CS(\Sigma)}\right) \quad (11)$$

where we assume a 3-sphere of radius  $L_P$  (Planck length) at the starting point,  $\text{vol}(\Sigma)$ ,  $CS(\Sigma)$  are the volume and Chern-Simons functional, respectively, which are topological invariants of the hyperbolic homology 3-sphere  $\Sigma$ . Therefore, we have to look for an expression of the mass which depends on the length scale. Lets start with the Planck mass

$$m_P = \sqrt{\frac{hc}{G}}$$

and made the following manipulations

$$m_P = \sqrt[6]{\frac{h^3 c^3}{G^3}} = \sqrt[3]{\frac{h^2}{G} \cdot \sqrt{\frac{c^3}{hG}}} = \sqrt[3]{\frac{h^2}{L_P \cdot G}}$$

to obtain a relation between length ( $L_P$ ) and mass ( $m_P$ ) (without using the Compton wave length). This relation can be generalized to

$$M(L) = \sqrt[3]{\frac{h^2}{L \cdot G}}$$

a length dependent mass scale. Together with (11) we obtain the formula

$$M = \sqrt{\frac{hc}{G}} \cdot \exp\left(-\frac{\text{vol}(\Sigma)}{2 \cdot CS(\Sigma)}\right)$$

Therefore, for a +8 Dehn-surgery with a special cusp (generating a geodesic of minimal length 0.3531), we obtain a homology 3-sphere  $\tilde{\Sigma}(8_{10})$  with

$$\begin{aligned} \text{vol}(\tilde{\Sigma}(8_{10})) &= 5,902827... \\ CS(\tilde{\Sigma}(8_{10})) &= 0.07546... \end{aligned}$$

and calculate the mass to

$$M_H \approx 126 \text{ GeV}$$

This result is not totally surprising, because there is an infinity of homology 3-spheres and it least one value should fit. The inflation process with theses values (putted into formula (11)) leads to around 117 of e-folds much larger than the minimal required value of 60 e-folds. Therefore if our assumption about a relation between inflation and the Higgs boson is correct, then one has to found experimental hints of an inflation with 117 e-folds of expansion.

This model is quite general, applicable also to the case of small exotic smooth  $\mathbb{R}^4$  where Casson handles can not be smoothly reduced to the 2-handles. Turning to the  $su(2)$  Lie algebra valued field  $\phi = \phi^a \cdot T^a$  and assuming the adjoint representation for such  $\phi$  and taking  $\phi^a \phi^a = v^2 \neq 0$ , one recovers, up to the additive and multiplicative constants, the appearance of the general shape of the Higgs potential as in (1). That is why the Higgs potential and field can be modeled by a Morse function on Casson handles of exotic  $\mathbb{R}^4$ , given a  $SU(2)$  YM theory on it. This is the smooth topological indication on magnetic monopoles emerging from exotic 4-geometry on  $\mathbb{R}^4$  which complements, and extends over non-BPS case, the static approach via Atiyah-Hitchin moduli space. At this stage, to avoid complications from the curvature of exotic  $\mathbb{R}^4$ , one considers the YM theory as on the flat  $\mathbb{R}^4$ , whereas the impact of exoticness is being the appearance of the Higgs potential from the Morse function. However, there exists yet another aspect of exotic smoothness which complements these just discussed and which appears as important for color confinement in  $SU(2)$  YM theories when on exotic  $\mathbb{R}^4$ . This is the smooth differentiability of some merely continuous functions on the standard  $\mathbb{R}^4$ . This is the topic of the next section.

#### IV. THE MECHANISM FOR GEOMETRIC CONFINEMENT AND THE DUAL MEISSNER EFFECT

Now we want to show that confinement in YM theories can be generated by the non-trivial geometry of the background where the theory is formulated. Here, our concern is the pure  $SU(2)$  YM on some exotic  $\mathbb{R}^4$ ,  $e$ . Note that the topology of  $e$  is the same as  $\mathbb{R}^4$  but its smoothness structure differs. Even though exotic  $\mathbb{R}^4$ s are non-flat Riemannian 4-manifolds, the direct action of their curvature is neglected, whereas various results derived from  $e$  become important. In the following we will make use of: i) Magnetic monopoles appearing from exotic 4-geometry in some limit of general relativity (Sec. II); ii) the scalar function, the Higgs potential, originated from the smooth topology of  $e$  which is the Morse function (Sec. III); iii) the appearance of the dominant abelian phase in pure  $SU(2)$  YM on the standard  $\mathbb{R}^4$ , i.e. the abelian projected effective gauge theory (APEGT) [4, 5], which agrees with the action of the Higgs from (ii) and the condensation of magnetic monopoles from (i) above.

Let us describe briefly the mechanism of confinement of electrically charged particles ('quarks') via the condensation of monopoles as in the dual Meissner effect. We begin with the 4-d relativistic generalization of the Landau-Ginzburg (GL) Lagrangian

$$S = - \int d^4x \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - |(\partial_\mu + ieA_\mu)\phi|^2 + \frac{\lambda}{4} (\phi\phi^* - v^2)^2 \right). \quad (12)$$

This is an abelian theory with  $U(1)$  gauge field and the complex Higgs field  $\phi$ . In the dual abelian Higgs theory  $\phi$  represents a magnetic charge condensate and  $F_{\mu\nu}$  is replaced by the dual  $\bar{F}_{\mu\nu}$  where the covariant derivative reads:  $D_\mu = \partial_\mu + igB_\mu$  and  $g$  is the magnetic charge.  $\bar{F}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + \bar{G}_{\mu\nu}$  where  $\bar{G}_{\mu\nu}$  is the usual string term allowing a non-zero electric current  $\partial_\mu G^{\mu\nu} = j^\nu$  and electric charges  $e$ . The Lagrangian is rewritten as:

$$S[B, \phi, \phi^*] = - \int d^4x \left( \frac{1}{4} \bar{F}_{\mu\nu} \bar{F}^{\mu\nu} - \frac{1}{2} (D_\mu \phi)(D_\mu \phi)^* + \frac{\lambda}{4} (\phi\phi^* - v^2)^2 \right). \quad (13)$$

This action is invariant with respect to the following abelian gauge transformations:

$$B_\mu \rightarrow B_\mu + \partial_\mu \beta, \quad \phi \rightarrow e^{-ig\beta} \phi.$$

Rewriting  $\phi$  as:  $\phi(x) = S(x)e^{ig\psi(x)}$ , where  $S(x)$  and  $\psi(x)$  are real, the action reads:

$$S[B, \psi, S] = \int d^4x \left( -\frac{1}{2} \bar{F}^2 + \frac{g^2 S^2}{2} (B + \partial\psi)^2 + \frac{1}{2} (\partial S)^2 - \frac{\lambda}{4} (S^2 - v^2)^2 \right). \quad (14)$$

The ground state is now achieved at  $S = v$ , so that the field  $S$  describes a 'Higgs' particle with mass  $m_H = 2v\sqrt{\lambda}$  and the gauge boson acquires mass  $m_B = gv$ .

Noticing the gauge symmetries of the action (14) which read:

$$B \rightarrow B + \partial\beta, \quad S \rightarrow S, \quad \psi \rightarrow \psi - \beta,$$

and rewriting  $S$  as  $S = v + s$  the equation of motion for  $\tilde{B}_\mu = B_\mu + \partial\psi$  at the lowest order in the interactions with  $s$ , becomes:

$$\partial^\mu \bar{F}_{\mu\nu} + g^2 v^2 \tilde{B}_\nu = 0.$$

This is equivalent to

$$(\square + m_B^2) \tilde{B}_\mu = 0, \quad \partial^\mu \tilde{B}_\mu = 0.$$

In this way it is explicitly seen that the  $\tilde{B}$  field (photon) acquires mass  $m_B$ , hence the field has a finite penetration depth into the dual superconducting medium which is defined as the inverse of the photon mass  $\Lambda = \frac{1}{m_B}$ . Another characteristic is the correlation length which is  $L = \frac{1}{m_H}$ . Magnetic monopoles can condense and the electric field is expounded from the superconductor and in case of a 2-nd kind (dual) superconductor ( $\Lambda > L$ ), the electric flux is quantized and forced to form (dual) Abrikosov lines, with the potential proportional to the distance between electric charges. This is more-or-less the idea of the confinement of electric charges via the magnetic monopoles condensation and the dual Meissner effect in the abelian YMH theory, through the action of the Higgs scalar. There are Wilson loop observables detecting the confining phase, i.e. one shows that the area law,  $W(C) \sim e^{-\text{Area}(C)}$ , is fulfilled for these operators  $W(C) = \langle P e^{i \oint_C dx^\mu B_\mu} \rangle$ . The potential, proportional to the separation of electric charges, follows. The contour  $C$  can be chosen as rectangular with 2 parallel edges one positively oriented with time (charged particle) and one in the reverse direction (antiparticle).

However, QCD, or our pure  $SU(2)$  YM case on Euclidean  $\mathbb{R}^4$  given by the Lagrangian (5), does not contain any scalar Higgs field and is not any abelian theory. Hence, the emergence of the Higgs field and its potential from exotic 4-geometry is a way of introducing them into the YM theory. The pure YM is also deformed to agree now with the effects of an exotic  $\mathbb{R}^4$  and Higgs from it. There are alternative mechanisms for the confinement in non-abelian theories without any scalar Higgs field included directly into the YM theory, like maximal abelian gauge (MAG) or abelian gauge projection. One shows that the choice of this abelian projection indeed gives rise to the effect of the confinement, but the procedure seemed to be gauge dependent. As shown by Kondo and his collaborators in a series of papers, the gauge independent mechanism for confinement in  $SU(2)$  and  $SU(3)$  YM theories emerges naturally when turning to the effective theory - APEGT. Also, as shown in Ref. [16] the MAG method can be formulated in a gauge independent way by making use of the non-abelian Bianchi identity. Thus, in the case of YM theory on an exotic  $\mathbb{R}^4$ , such that Higgs field and potential can be generated by the exotic background, the resulting YM theory should become a kind of effective APEGT theory. In this way the agreement of YM theory without Higgs, with the effects of the exotic background where Higgs field comes from, can be achieved. That is why we claim that pure  $SU(2)$  YM on  $e$  agrees with APEGT on  $\mathbb{R}^4$  in the limit where Higgs field and magnetic monopoles decouple from exotic 4-geometry. One way to see this result is to show that the effective Higgs field (the scalar Stückelberg field in APEGT) and magnetic monopole current, are generated by the exotic geometry of  $e$ . Next, given the bare possibility that YM theory can be formulated on a smooth  $e$  we arrive at the conclusion that YM on  $e$  should be twisted like APEGT. The effects of exotic geometry of  $e$  are represented by the appearance of the Higgs field and the magnetic current. On the other hand, APEGT directly predicts confinement, so we infer that the geometry of  $e$  causes the confinement in YM on  $e$ .

APEGT is a version of an effective theory derived from maximal abelian projection or gauge in YM theory. The MAG in case of  $SU(2)$  YM theory is performed via the projection onto the maximal weight  $\phi_0^3 = \frac{\sigma_3}{2}$  and in the representation where it is diagonal. The generators of  $SU(2)$  are as usual  $T^a = \frac{1}{2}\sigma^a$ ,  $a = 1, 2, 3$  and  $\sigma^a$  are Pauli matrices. In a general case, an abelian projection on a field  $\phi = \vec{\phi} \cdot \vec{\sigma} = \phi^a \sigma^a$  in the adjoint of  $SU(2)$ , is defined via the following procedure. Gauge invariant projected field strength (the non-abelian 't Hooft tensor) is given by:

$$F_{\mu\nu}^{(\phi)} = \hat{\phi} \cdot F_{\mu\nu} - \frac{1}{g} \hat{\phi} \cdot (D_\mu \hat{\phi} \wedge D_\nu \hat{\phi}) = \hat{\phi}^a F_{\mu\nu}^a - \frac{1}{g} \hat{\phi}^a (D_\mu \hat{\phi}^a \wedge D_\nu \hat{\phi}^a)$$

where  $\hat{\phi} = \frac{\vec{\phi}}{|\phi|}$  is the vector of  $\phi$  in the iso-space and  $D_\mu \hat{\phi} = (\partial_\mu - g A_\mu \wedge) \hat{\phi}$ . The projected field strength  $F_{\mu\nu}^{(\phi)}$  is thus rewritten as:

$$F_{\mu\nu}^{(\phi)} = \partial_\mu (\hat{\phi}^a A_\nu^a) - \partial_\nu (\hat{\phi}^a A_\mu^a) - \frac{1}{g} \hat{\phi}^a (\partial_\mu \hat{\phi}^a \wedge \partial_\nu \hat{\phi}^a).$$

Fixing  $(\hat{\phi}^a) = (0, 0, 1)$  the result is

$$F_{\mu\nu}^{(3)} = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3$$

which is the abelian projection on  $\hat{\phi}$  and possesses the residual  $U(1)$  symmetry given by the rotation around  $\hat{\phi}$ . Magnetic monopoles can exist whenever there is a non-zero dual current  $j_\nu = \partial^\mu \star F_{\mu\nu}$ .

In the  $SU(2)$  YM theory the non-zero current  $J_\mu = D_\nu \star F_{\mu\nu}$  where  $\star F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$  is the ordinary dual field strength of the gauge potential  $A$ , directly violates the non-abelian Bianchi identity, i.e.  $D_\nu \star F_{\mu\nu} \neq 0$ . As shown in Ref. [16] the projected abelian magnetic current  $j_\mu$  is proportional to  $J_\mu$ , i.e.

$$\partial_\mu \star F_{\mu\nu}^{(\phi)a} = \text{Tr}(\phi^a J_\nu)$$

where now  $\star F_{\mu\nu}^{(\phi)a}$  is the dual 't Hooft tensor. As a result, the violation of the Bianchi identity is the sufficient and necessary condition for the appearance of the magnetic current in solutions of YMH theory. The abelian 't Hooft tensor  $f_{\mu\nu}$  is assigned to the abelian projection  $\phi^3$  as the abelian field strength of the residual  $U(1)$  gauge group.

To understand more clearly the reason behind the confinement in  $SU(2)$  YM theory on exotic  $\mathbb{R}^4$  one looks at the singularity of  $A_\mu^3(x)$  which is the source for magnetic currents after the abelian projection as above. Then, one can make use of the (double-charged abelian) Wilson loop and it is shown that the usual area law holds for them, i.e.  $W_2(C) = \left\langle P e^{2i \oint_C dx^\mu A_\mu^3 \frac{\sigma_3}{2}} \right\rangle$  and  $W_2(C) \sim e^{-\text{Area}(C)}$ .

The abelian-projected effective gauge theory of Kondo is formed when in pure, say  $SU(2)$  YM theory on  $\mathbb{R}^4$ , the maximal abelian projection is performed and one integrates out the off-diagonal fields. Thus, one chooses the maximal abelian gauge leading to the confinement in this effective theory. As we saw MAG relies on the choice of the gauge-dependent quantity  $Z(x) = Z^a(x)T^a$  which transforms in the adjoint representation of  $SU(2)$ , i.e.  $Z'(x) = U(x)Z(x)U^\dagger(x)$ . Next, one diagonalizes  $Z(x)$  by the gauge rotation, so that the eigenvalues appear as  $\lambda_i(x)$ ,  $i = 1, 2$  on the diagonal of the transformed  $Z(x) = \text{diag}(\lambda_1(x), \lambda_2(x))$ .

Whenever  $\lambda_i(x) = \lambda_j(x)$ ,  $i \neq j$  for  $x \in \mathbb{R}^4$  the singularity appears which is the 't Hooft magnetic monopole solution. In fact, the gauges leaving the procedure, i.e.  $Z(x)$ , invariant reduces now to the  $U(x) = \text{diag}(e^{i\theta_1}, e^{i\theta_2})$  such that  $\theta_1 + \theta_2 = 0$ . But these gauges reduce  $SU(2)$  YM theory to the  $U(1)$  abelian theory. More explicitly, according to the above prescription, the gauge vector potential is decomposed into the diagonal  $U(1)$  residual part (the maximal torus) and the off-diagonal  $SU(2)/U(1)$ :

$$A_\mu(x) = \sum_{a=1}^3 A_\mu^a(x)T^a = a_\mu(x)T^3 + \sum_{a=1}^2 A_\mu^a(x)T^a.$$

One decomposes the usual field strength  $F_{\mu\nu}T^a = (\partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) - i[A_\mu^a, A_\nu^a])T^a$  as:

$$F_{\mu\nu}T^a = [f_{\mu\nu}(x) + C_{\mu\nu}(x)]T^3 + S_{\mu\nu}^2T^2 + S_{\mu\nu}^1T^1$$

where again the summation on  $a = 1, 2, 3$  is assumed. Here [4]:

$$\begin{aligned} f_{\mu\nu}(x) &= \partial_\mu a_\nu(x) - \partial_\nu a_\mu(x), \\ S_{\mu\nu}^a(x) &= \partial_\mu \delta^{ab} A_\nu^b - \epsilon^{ab3} a_\mu A_\nu^b - [\partial_\nu \delta^{ab} A_\mu^b - \epsilon^{ab3} a_\nu A_\mu^b], \quad a, b = 1, 2 \\ C_{\mu\nu}(x)T^3 &= -i[A_\mu(x), A_\nu(x)], \end{aligned}$$

so that the diagonal part reads:

$$F_{\mu\nu}^3 = f_{\mu\nu} + A_\mu^1 A_\nu^2 - A_\nu^2 A_\mu^1.$$

The pure YM action (5) can be rewritten in terms of the diagonal field, as:

$$S_{YM} = -\frac{1}{4g^2} \int d^4x [(f_{\mu\nu} + C_{\mu\nu})^2 + (S_{\mu\nu}^a)^2].$$

One introduces the dual tensor field to the diagonal  $F_{\mu\nu}^3$  as:

$$B_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^3 = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (f_{\rho\sigma} + C_{\rho\sigma})$$

and its Hodge decomposition gives:

$$B_{\mu\nu} = b_{\mu\nu} + \tilde{\chi}_{\mu\nu}, \quad b_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu, \quad \tilde{\chi}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} (\partial^\rho \chi^\sigma - \partial^\sigma \chi^\rho). \quad (15)$$



Introducing the dual  $\tilde{f}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}f^{\rho\sigma}$  the (gauge fixed) YM action reads:

$$\overline{S}_{YM} = \int d^4x \left[ -z_a \frac{1}{g^2} f_{\mu\nu} f^{\mu\nu} - z_b \cdot g^2 (b_{\mu\nu} b^{\mu\nu} + \tilde{\chi}_{\mu\nu} \tilde{\chi}^{\mu\nu}) + \frac{1}{2} z_c b_{\mu\nu} \tilde{f}_{\mu\nu} + \frac{1}{2} z_c \chi_{\mu\nu} f_{\mu\nu} + \dots \right]$$

which, after integrating out  $\chi$  and neglecting the ghost self-interaction and some higher derivatives, becomes the effective theory written in terms of the abelian  $a_\mu$  and the dual  $b_\mu$ :

$$S = \int d^4x \left[ -z_a \frac{1}{g^2} f_{\mu\nu} f^{\mu\nu} + i\bar{c}^a D_\mu^{ac}[a] D_\mu^{cb}[a] c^b - z_b \cdot g^2 b_{\mu\nu} b^{\mu\nu} - z_c b_\mu k^\mu \right].$$

Here  $D_\mu[a]^{ab} = \partial\delta^{ab} - \epsilon^{ab3}a_\mu$  as before and  $c^a, \bar{c}^a$  are ghost, anti-ghost fields due to the gauge fixing functional as in [4]. We do not make use of the explicit form of the expressions  $z_a, z_b, z_c$  appearing here and the interested reader can find them in [4]. The crucial point is the presence of the magnetic current  $k_\mu = \partial^\nu \tilde{f}_{\mu\nu}$  of the dual field strength  $\tilde{f}_{\mu\nu}$ . The confinement via the dual Meissner effect appears whenever the diagonal  $a_\mu$  is singular - in this case the magnetic current  $k_\mu$  is non-zero and it couples to the dual  $b_\mu$ . We propose here the purely geometric way of generating the singularity of the diagonal potential  $a_\mu$ . Let us suppose that the YM theory is formulated on an *exotic*  $\mathbb{R}^4$  in a specific way. Namely, the diagonal components of the dual vector potential  $A_\mu$  propagates on exotic geometry while the off-diagonal ones are not sensitive to the exoticness, hence propagate on the standard  $\mathbb{R}^4$ . Thus  $a_\mu$  propagates on exotic  $\mathbb{R}^4$  either. Next, let us make use of the fundamental difference between standard and exotic smoothness on  $\mathbb{R}^4$ : (i) *Smooth functions in the standard and exotic structures are not the same, i.e. there should exist continuous non-differentiable functions in the standard smooth structure which are smooth in an exotic structure on  $\mathbb{R}^4$ .* Let  $A_\mu(x)$  be such a function. Additionally we should argue that singularities in  $A_\mu$  leading to the confinement are of the kind generated by the change of the smoothness as above. To this end let us consider the abelian projected field strength  $f_{\mu\nu}$  as given directly in terms of the singular gauge potential  $A_\mu$ :

$$f_{\mu\nu} = \text{Tr}(ig[A_\mu, A_\nu]T^3) + \text{Tr}\left(\frac{i}{g}U(x)[\partial_\mu, \partial_\nu]U^\dagger(x)T^3\right) \quad (16)$$

where  $U(x)$  is a gauge transformation. The magnetic current of the magnetic monopoles derived from  $f_{\mu\nu}$  reads:  $k_\mu = \frac{1}{2}\epsilon_{pq\mu s}\partial^s f_{pq}$  which is equivalent to the contribution from the first term on the RHS of (16), i.e.  $K_\mu = \frac{1}{2}\epsilon_{pq\mu s}\partial^s(g\epsilon^{ab3}A_p^a A_q^b)$ . The singularity of  $A$  can be developed by the abelian gauge-fixing conditions put on the transformed gauge potential  $A'_\mu(x) = U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x)$ . The abelian gauge field  $a_\mu = \text{Tr}(T^3 A_\mu)$  corresponding to  $A$  develops a singularity under the gauge transformation  $U(x)$ , too. The corresponding transformation for the field strength reads:

$$F'_{\mu\nu}(x) = \partial_\mu A'_\nu(x) - \partial_\nu A'_\mu(x) - ig[A'_\mu(x), A'_\nu(x)]$$

such that the corresponding abelian field strength is:

$$f_{\mu\nu} = \partial_\mu a'_\nu - \partial_\nu a'_\mu = \text{Tr}[T^3(U(x)F_{\mu\nu}U^\dagger(x) + ig[A'_\mu, A'_\nu])].$$

For the magnetic current  $k_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu f^{\rho\sigma}$ , only when  $U(x)$  is singular the term  $\frac{i}{g}U\partial_\mu U^\dagger$  in the expression for  $A'_\mu(x)$ , gives the non-vanishing contribution to  $k_\mu$ . As shown in Ref. [4] the contributions to  $k_\mu(x)$  derived from this singular  $A_\mu(x)$  are of two kinds: first derived from the first term in the RHS of (16) and it is a magnetic monopole sitting at  $\vec{r} = 0$ , and the second, corresponding to the second term in the RHS in (16), is the 'Dirac string'. Both contributions have distributional character. Namely, the calculation of the contribution from the magnetic monopole part from the singular  $A_\mu(x)$ , gives the magnetic current as [4]:

$$k_\mu = \frac{4\pi}{g}\delta_{\mu 0}\delta^{(3)}(x).$$

Precisely this kind of contributions allows for the appearance of the effects of magnetic monopoles, hence also confinement, in YM theory. Again, the confinement can be detected via Wilson loops observables and it is shown that the usual area law holds true also in this case.

One could wonder whether such singularities (contributions) might appear as the result of the change of smoothness on the background  $\mathbb{R}^4$ . From general considerations on distribution theory, one infers that every tame distribution is derivable from some continuous functions by taking their distributional derivatives and combinations thereof. Given a tempered distribution  $h$  one can always find a continuous slowly increasing function  $H$ , such that  $h = D^{|\alpha|}H$  for some multiindex  $|\alpha|$ . Similar results hold true also for more general classes of distributions. So, starting with continuous functions and performing distributional derivatives gives rise to general distributions.

Turning to exotic smooth structures on  $\mathbb{R}^4$  which would be different non-diffeomorphic ones, merely continuous functions in one structure which are smoothly differentiable in the other, have to exist. Otherwise, the structures would be diffeomorphic. The results of Ref. [17] show that a 'general singularity' emerges as some distribution in the process of changing the smoothness on  $\mathbb{R}^4$  to the standard one. One finds a continuous function which is smoothly differentiable as exotic smooth function, and by the distributional differentiation, after (possibly) infinite many differentiations, the suitable singularity emerges. Moreover, there are infinite (continuum) many different small smooth  $\mathbb{R}^4$ s in the fixed radial family, hence from the general point of view there is a good chance to model the required distributional singularity by some exotic structure from the family.

However, there are also direct reasons explaining why the non-zero magnetic current appears in YM theory on exotic  $\mathbb{R}^4$  after the change of smoothness into the standard one. Let us consider  $a_\mu(x)$  and  $A_\mu(x)$  as propagating on some exotic  $\mathbb{R}^4$  from the fixed radial family. As we know the characteristic feature of all such small  $\mathbb{R}^4$  is that they determine codimension-one foliations of  $S^3$  with non-zero GV invariant related to the radius of the family. We make use of this non-zero GV invariant directly:

$$\text{GV} \neq 0 \text{ means that } 0 \neq [d\theta \wedge \theta] \in H^3(S^3, \mathbb{R})$$

where  $\theta$  is the one form derived from the integrability condition of the 1-form  $\omega$  for any codimension-one foliation, i.e.  $d\omega \wedge \omega = 0$  and  $d\theta = -\theta \wedge \omega$ . The differential forms here can be considered as defined on the separating 3-manifold  $Y_n$  or on 3-sphere  $S^3$  from which  $Y_n$  is obtained via the surgery along some knot [1]. Such realization of the non-zero GV from exotic 4-space is a global effect derived from an exotic  $\mathbb{R}^4$ . Extending 1-forms  $\theta$  over the 4-region still has to give nontrivial GV number when restricting the form on  $Y_n$  and integrating over it.

This observation suggests that the restricted abelian vector potential  $a_\mu^3(x)|_{Y_n}$  and after the projection on the standard  $\mathbb{R}^4$  one can take a certain 1-form proportional to  $\theta$ . However,  $Y_n$  is embedded in the exotic  $\mathbb{R}^4$  and

$$0 \neq \text{GV}_{Y_n} \in H^3(Y_n, \mathbb{R}), \text{ so that } d\theta \sim f_{\mu\nu}|_{Y_n} \text{ and } 0 \neq \epsilon_{\nu pr} \partial^\nu f_{|Y_n}^{pr} \in H^3(Y_n, \mathbb{R}).$$

As the result

$$k_{\mu|Y_n} = \frac{1}{2} \epsilon_{\mu\nu pr} \partial^\nu f_{|Y_n}^{pr} \neq 0.$$

Moreover,  $S^3$  is embedded in the standard  $\mathbb{R}^4$  and one can always choose the codimension-one foliation of  $S^3$  such that  $0 \neq \text{GV}_{S^3} = \text{GV}_{Y_n}$  and work with  $H^3(S^3, \mathbb{R})$  in the standard  $\mathbb{R}^4$  [1]. That is why

$$\text{GV}_{S^3}[S^3] \sim \frac{1}{2} \int_{S^3} dv \epsilon_{\mu\nu pr} \partial^\nu f_{|S^3}^{pr} \neq 0.$$

The extension of the nontrivial class (non-exact 3-form) in  $H^3(S^3, \mathbb{R})$  to a 4-dimensional region gives rise to the singular expressions for the magnetic current and vector potential. On the other hand, a given standard  $\mathbb{R}^4$  and a vector potential  $a_\mu(x)$  propagating on it, one has  $\text{GV} = 0$  and  $k_\mu = 0$ , i.e. no nontrivial magnetic current exists for these global non-singular vector bosons.

Consider in the following that the singularity in  $a_\mu$  or  $A_\mu$  is the result of changing the smoothness from exotic to the standard  $\mathbb{R}^4$ . The diagonal vector fields propagate on exotic smooth geometry on  $\mathbb{R}^4$  and they are exotic smooth and become singular when the smoothness is changed into standard one. As the result, the non-trivial magnetic current emerges. The theory shows confinement when additionally the effects are required to be compatible with the non-trivial effective Higgs potential, hence  $v \neq 0$ , derived from the exotic handle-body. To explain it qualitatively let us note that in the APEGT the effective abelian magnetic monopole current, reads:

$$K^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \partial_\nu (\epsilon^{ab3} A_\rho^a A_\sigma^b),$$

as we observed already and which is the contribution of the first term in the RHS of (16). The dual magnetic field  $b_\mu$  stays massless whenever the magnetic  $U(1)$  symmetry is not broken. This result follows from extracting the  $b_\mu$ -dependent parts of the effective action of APEGT [4]. The non-zero mass for  $b_\mu$  breaks the  $U(1)$  symmetry so that the compact correlation function for the magnetic current becomes

$$\langle K_\mu K_\nu \rangle = g^2 \delta_{\mu\nu} \delta^{(4)}(x - y) f(x) + \dots \quad (17)$$

and the mass term for the dual field  $b_\mu$  appears in the action for this field, as

$$S[b] = \int d^4x \left[ -\frac{1}{4} b_{\mu\nu} b^{\mu\nu} + \frac{1}{2} g^2 f(x) b_\mu(x) b_\mu(x) + \dots \right] \quad (18)$$

where  $f(x) = m_b^2$  is the square of the mass, and  $b_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$  as in (15). This is the dual Meissner effect for the APEGT without the Higgs field where the dual gauge field acquires the non-zero mass due to the condensation of magnetic monopoles as in (17).

However, the 'Higgs' scalar  $\phi(x)$  can be reintroduced into the theory as the so called Stückelberg field. Again following [4], taking  $\phi(x) = \frac{m_b}{\sqrt{2}} e^{i\theta(x)}$ , the mass term  $\frac{1}{2}m_b^2 b_\mu(x)b_\mu(x)$  is rewritten as

$$\frac{1}{2}m_b^2(b_\mu(x) - \partial_\mu\theta(x))^2 = |(\partial_\mu - ib_\mu(x))\phi(x)|^2$$

which is  $U(1)$ -invariant. It appears that our dual abelian gauge theory given by  $S[b]$  in (18) is now equivalent to the dual Ginzburg-Landau (GL) theory, i.e. the dual abelian Higgs theory (12):

$$S[b] = \int d^4x \left[ \frac{-1}{4} b_{\mu\nu} b^{\mu\nu} + |(\partial_\mu - ig^{-1}b_\mu(x))\phi(x)|^2 + \lambda(|\phi(x)|^2 - \frac{m_b^2}{2})^2 + \dots \right].$$

This is best seen by taking the so called London limit in this GL theory where also the coupling constant is reversed, i.e.  $\lim_{\lambda \rightarrow \infty} \lambda(|\phi|^2 - \frac{m_b^2}{2})^2$  and  $g \rightarrow \frac{1}{g}$ . Now, in analogy with the abelian YMH theory, the deconfining phase corresponds to the  $m_b = 0$  and the confinement appears whenever the minimum of the potential  $V(\phi)$  is non-zero, hence the  $m_b \neq 0$ . This last case, however, was derived without the Higgs scalar in  $SU(2)$  YM theory on an exotic  $\mathbb{R}^4$ . Besides, it is seen that the Higgs potential generated from the exotic handle-body appears as the dual potential in the abelian YMH theory derived from the dual APEGT theory. This result has its reasoning in the requirement that magnetic monopoles effects from exotic  $\mathbb{R}^4$  agree with the Higgs potential emerging from the exotic handle-body. Certainly, one can detect the confinement also by the suitable Wilson line operators in APEGT [4].

Therefore we obtain the geometric mechanism for the confinement in  $SU(2)$  YM theory on certain small exotic  $\mathbb{R}^4$ s from the fixed radial family. The APEGT can be considered as the effective theory matching with the exotic 4-geometry and exhibiting the confinement when YM theory is formulated, in the suitable way presented here, on an exotic  $\mathbb{R}^4$ . For the smooth fields propagating in the standard smooth structure on  $\mathbb{R}^4$  no geometric singularity will appear, hence no geometric magnetic monopoles for these vector bosons exist and the theory do not show confinement.

## V. DISCUSSION

In this paper we show that certain differential structures on  $\mathbb{R}^4$ , which are nonstandard smoothness structures, can push forward our understanding of the confining phase of 4-d  $SU(2)$  YM theory. As a rule the structures are grouped in the DeMichelis-Freedman type radial family of smooth exotic  $\mathbb{R}^4$ s. Since there is still disagreement about the pure YM theory structure of the vacuum and the origin of the confinement from the asymptotically free infrared part of the theory, our proposal gives some new geometric perspective on these problems. The scenario advocated here can be described as follows. In the infrared limit all gluons propagate on the standard 4-geometry, while in sufficiently low energies some (diagonal, dual) modes propagate on an exotic  $\mathbb{R}^4$ . This is the geometric reason for breaking the non-abelian gauge symmetry and performing the abelian projection. Thus, we find the reason for the abelian projection rather in the outside geometry of the background, than in the YM theory itself.

Further breaking of the remnant dual  $U(1)$  symmetry is also explained via fake smoothness of  $\mathbb{R}^4$ . The exotic geometry decouples from the theory in sufficiently high energies and the deconfining phase is reached. This specific relation (interaction) of geometry and YM theory is possible only in dimension four.

Magnetic monopoles appear in the theory when fake 4-geometry is broken to the standard one. Equivalently, exotic 4-geometry gives rise to the magnetic current and Higgs potential in YM theory on  $\mathbb{R}^4$ . Fixing the radial family of small exotic  $\mathbb{R}^4$ s gives the codimension-1 foliations of some compact 3-manifolds:  $Y_n$  embedded in the exotic  $\mathbb{R}^4$ , and  $S^3$  in the standard  $\mathbb{R}^4$ . Both foliations have the same, non-zero value of the Godbillon-Vey (GV) invariant [3]. We have shown in a direct way that the confinement in YM theories is caused by the exoticness of open background 4-manifolds where the GV class breaks the dual magnetic abelian symmetry. This idea is completely new approach to the confinement in YM theories. The theory, which effectively encompasses the effects from exotic geometry and those of deformed YM theory, is chosen as APEGT of Kondo. Some other choices are still possible. Recent proposal for the analytical generation of confinement, is to reformulate YM theory in terms of new variables and then to apply the dual Meissner effect [16, 18, 19].

Several points require further comments. In the paper we considered the monopoles condensation model for confinement in the  $SU(2)$  YM theory on exotic  $\mathbb{R}^4$  where monopoles are generated by this fake geometry. Other mechanisms for confinement, like Gribov-Zwanziger and Dyson-Schwinger propagators, vacuum wave-functions or gluon chains [20], are also likely to contain elements derivable from exotic 4-geometry. Even though, they are not directly accessible due to the entirely unknown analytical shape of the global exotic geometry on  $\mathbb{R}^4$ , their conjectural agreement with the proposal, would serve as a test for it. In our case, magnetic monopoles, Higgs potential, breaking of the dual

magnetic  $U(1)$  symmetry and abelian projection from non-abelian gauge theory, all these effects are derivable from general properties of the radial family and handle-body smooth topology of the exotic  $\mathbb{R}^4$ .

The extension of the  $SU(2)$  YM theory into the more physical  $SU(3)$  case and including the matter (quarks) is possible. This part of the theory will be performed separately, however, the important ingredient would presumably be the topology of some  $S^2$  fibrations which makes the  $SU(3)$  case rather exceptional. Again, the propagation of some fields (not all) in exotic geometry of the background  $\mathbb{R}^4$  would be the reason for the appearance of the abelian projection and confinement. Again, exotic 4-geometry of the background can break the dual  $U(1)$  magnetic symmetry. The deeper discussion of the quantum level of YM theories on exotic  $\mathbb{R}^4$  is also an important challenge and can be approached from various perspectives which sheds further light on confinement.

Wilson loops display the confinement in the YM theory. If the confinement comes from an exotic  $\mathbb{R}^4$ , certain Wilson loops observables could be further used for detecting the exotic smoothness on background  $\mathbb{R}^4$ . The fundamental question which emerges here, is whether one can propose specific observables in 'real' YM theory and specific methods to measure them which would indicate, also experimentally, the presence of an exotic background for  $SU(3)$  YM theory with matter. This point becomes more important because QCD non-perturbative calculations show severe limitations from the fundamental point of view. In this case, the connection of exotic 4-geometry with some states of 'effective' matter, claimed some time ago, would not be only a bold formal guess [21, 22]. After all, breaking the non-abelian gauge, and the remnant magnetic, symmetries due to exotic smooth structures, could leave specific experimental trace. To identify it properly requires, however, a careful analysis and further insights.

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